

# General temporal instability criteria for stably stratified inviscid flow

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The temporal instability of stably stratified flow was investigated by analyzing the Taylor-Goldstein equation theoretically. According to this analysis, the stable stratification  $N^2 \geq 0$  has a destabilization mechanism, and the flow instability is due to the competition of the kinetic energy with the potential energy, which is dominated by the total Froude number  $Fr_t^2$ . Globally,  $Fr_t^2 \leq 1$  implies that the total kinetic energy is smaller than the total potential energy. So the potential energy might transfer to the kinetic energy after being disturbed, and the flow becomes unstable. On the other hand, when the potential energy is smaller than the kinetic energy ( $Fr_t^2 > 1$ ), the flow is stable because no potential energy could transfer to the kinetic energy. The flow is more stable with the velocity profile  $U'/U''' > 0$  than that with  $U'/U''' < 0$ . Besides, the unstable perturbation must be long-wave scale. Locally, the flow is unstable as the gradient Richardson number  $Ri > 1/4$ . These results extend the Rayleigh's, Fjørtoft's, Sun's and Arnol'd's criteria for the inviscid homogenous fluid, but they contradict the well-known Miles-Howard theorem. It is argued here that the transform  $F = \phi/(U - c)^n$  is not suitable for temporal stability problem, and that it will lead to contradictions with the results derived from the Taylor-Goldstein equation. However, such transform might be useful for the study of the Orr-Sommerfeld equation in viscous flows.

## 1. Introduction

The instability of the stably stratified shear flow is one of main problems in fluid dynamics, astrophysical fluid dynamics, oceanography, meteorology, etc. Although both pure shear instability without stratification and statical stratification instability without shear have been well studied, the instability of the stably stratified shear flow is still mystery.

On the one hand, the shear instability is known as the instability of vorticity maximum, after a long way of investigations (Rayleigh 1880; Fjørtoft 1950; Arnold 1965; Sun 2007, 2008). It is recognized that the resonant waves with special velocity of the concentrated vortex interact with flow for the shear instability (Sun 2008). Other velocity profiles are stable in homogeneous fluid without stratification. On the other hand, Rayleigh (1883) proved out that buoyancy is a stabilizing effect in the statical case. Thus, it is naturally believed that the stable stratification do favor the stability (see, e.g. Taylor 1931; Chandrasekhar 1961), which finally results in the well known Miles-Howard

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theorem (Miles 1961, 1963; Howard 1961). According to this theorem, the flow is stable to perturbations when the Richardson number  $Ri$  (ratio of stratification to shear) exceeds a critical value  $Ri_c = 1/4$  everywhere. In three-dimensional stratified flow, the corresponding criterion is  $Ri_c = 1$  (Abarbanel *et al.* 1984).

However, the stabilization effect of buoyancy is a illusion. In a less known paper, Howard & Maslove (1973) had shown with several special examples that stratification effects can be destabilizing due to the vorticity generated by non-homogeneity, and the instability depends on the details of the velocity and density profiles. One instability is called as Holmboe instability (Holmboe 1962; Ortiz *et al.* 2002; Alexakis 2009). Then Howard & Maslove (1973) stated three main points from the examples without any further proof. (a) Stratification may shift the band of unstable wave numbers so that some which are stable at homogeneous cases become unstable. (b) Conditions ensuring stability in homogeneous flow (such as the absence of a vorticity maximum) do not necessarily carry over to the stratified case, so that 'static stability' can destabilize. (c) New physical mechanisms brought in by the stratification may lead to instability in the form of a pair of growing and propagating waves where in the homogeneous case one had a stationary wave.

Recall the points by Howard & Maslove (1973), and that there is a big gap between Rayleigh's criterion and Miles-Howard's criterion, Yih (1980) even wrote "Miles' criterion for stability is not the nature generalization of Rayleigh's well-known sufficient condition for the stability of a homogeneous fluid in shear flow". The mystery of the instability is still cover for us.

Following the frame work of Sun (2007, 2008), this study is an attempt to clear the confusion in theories. We find that the flow instability is due to the competition of the kinetic energy with the potential energy, which is dominated by the total Froude number  $Fr_t^2$ . And the unexpected assumption in Miles-Howard theorem leads the contradiction to other theories.

## 2. General Unstable Theorem For Stratified Flow

### 2.1. Taylor-Goldstein Equation

The Taylor-Goldstein equation for the stratified inviscid flow is employed (Howard 1961; Yih 1980; Baines & Mitsudera 1994; Criminale *et al.* 2003), which is the vorticity equation of the disturbance (Drazin & Reid 2004). Considering the flow with velocity profile  $U(y)$  and the density field  $\rho(y)$ , and the corresponding stability parameter  $N$  (the Brunt-Vaisala frequency),

$$N^2 = -g\rho'/\rho, \quad (2.1)$$

where  $g$  is the acceleration of gravity, the single prime  $'$  denotes  $d/dy$ , and  $N^2 > 0$  denotes a stable stratification. The vorticity is conserved along pathlines. The streamfunction perturbation  $\phi$  satisfies

$$\phi'' + \left[ \frac{N^2}{(U-c)^2} - \frac{U''}{U-c} - k^2 \right] \phi = 0, \quad (2.2)$$

where  $k$  is the real wavenumber and  $c = c_r + ic_i$  is the complex phase speed and double prime  $''$  denotes  $d^2/dy^2$ . For  $k$  is real, the problem is called temporal stability problem. The real part of complex phase speed  $c_r$  is the wave phase speed, and  $\omega_i = kc_i$  is the growth rate of the wave. This equation is subject to homogeneous boundary conditions

$$\phi = 0 \text{ at } y = a, b. \quad (2.3)$$

It is obvious that the criterion for stability is  $\omega_i = 0$  ( $c_i = 0$ ), for that the complex conjugate quantities  $\phi^*$  and  $c^*$  are also physical solutions of Eq.(2.2) and Eq.(2.3).

Multiplying Eq.(2.2) by the complex conjugate  $\phi^*$  and integrating over the domain  $a \leq y \leq b$ , we get the following equations

$$\int_a^b [|\phi'|^2 + k^2|\phi|^2 + \frac{U''(U - c_r)}{|U - c|^2}|\phi|^2] dy = \int_a^b \frac{(U - c_r)^2 - c_i^2}{|U - c|^4} N^2 |\phi|^2 dy. \quad (2.4)$$

and

$$c_i \int_a^b [\frac{U''}{|U - c|^2} - \frac{2(U - c_r)N^2}{|U - c|^4}] |\phi|^2 dy = 0. \quad (2.5)$$

In the case of  $N^2 = 0$ , Rayleigh (1880) used Eq.(2.5) to prove that a necessary condition for inviscid instability is  $U''(y_s) = 0$ , where  $y_s$  is the inflection point and  $U_s = U(y_s)$  is the velocity at  $y_s$ . Using Eq.(2.5), Synge (1933) also pointed out that a necessary condition for instability is that  $U'' - \frac{2(U - c_r)N^2}{|U - c|^2}$  should change sign. But such condition is useless as there are two unknown parameters  $c_r$  and  $c_i$ .

As a first step in our investigation, we need to estimate the ratio of  $\int_a^b |\phi'|^2 dy$  to  $\int_a^b |\phi|^2 dy$ . This is known as the Poincaré's problem:

$$\int_a^b |\phi'|^2 dy = \mu \int_a^b |\phi|^2 dy, \quad (2.6)$$

where the eigenvalue  $\mu$  is positively definite for any  $\phi \neq 0$ . The smallest eigenvalue value, namely  $\mu_1$ , can be estimated as  $\mu_1 > (\frac{\pi}{b-a})^2$  (Mu *et al.* 1994; Sun 2007).

## 2.2. General Instability Theorem

In departure from previous investigations, we shall investigate the stability of the flow by using Eq.(2.4) and Eq.(2.6). As  $\mu$  is estimated with boundary so the criterion is global. We will also adapt a different methodology. If the velocity profile is unstable ( $c_i \neq 0$ ), then the equations with the hypothesis of  $c_i = 0$  should result in contradictions in some cases. Following this, a sufficient condition for instability can be obtained.

Firstly, substituting Eq.(2.6) into Eq.(2.4), we have

$$c_i^2 \int_a^b \frac{g(y)}{|U - c|^2} |\phi|^2 dy = - \int_a^b \frac{h(y)}{|U - c|^2} |\phi|^2 dy, \quad (2.7)$$

where

$$g(y) = \mu + k^2 + \frac{2N^2}{|U - c|^2}, \text{ and} \quad (2.8a)$$

$$h(y) = (\mu + k^2)(U - c_r)^2 + U''(U - c_r) - N^2. \quad (2.8b)$$

It is noted that  $g(y) > 0$  for  $N^2 \geq 0$ . Then  $c_i^2 > 0$  if  $h(y) \leq 0$  throughout the domain  $a \leq y \leq b$  for a proper  $c_r$  and  $k$ . Obviously,  $h(y)$  is a monotone function of  $k$ : the smaller  $k$  is, the smaller  $h(y)$  is. When  $k = 0$ ,  $h(y)$  has the smallest value.

$$h(y) = N^2 [\frac{(U - c_r)^2}{N^2/\mu} + \frac{U''(U - c_r)}{N^2} - 1]. \quad (2.9)$$

If we define shear, parallel and Rossby Froude numbers  $Fr_t$ ,  $Fr_s$  and  $Fr_r$  as

$$Fr_t^2 = Fr_s^2 + Fr_r^2, Fr_s^2 = \frac{(U - c_r)^2}{N^2/\mu}, Fr_r^2 = \frac{U''(U - c_r)}{N^2}, \quad (2.10)$$

where the shear Froude number  $Fr_s$  is a dimensionless ratio of kinetic energy to potential energy. As  $U''$  plays the same role of  $\beta$  effect in the Rossby wave (Sun 2006a, 2007), the shear Froude number  $Fr_r$  is a dimensionless ratio of Rossby wave kinetic energy to potential energy. Then  $h(y) \leq 0$  equals to  $Fr_t^2 \leq 1$ . Thus a general theorem for instability can be obtained from the above notations.

Theorem 1: If velocity  $U$  and stable stratification  $N^2$  satisfy  $h(y) \leq 0$  or  $Fr_t^2 \leq 1$  throughout the domain for a certain  $c_r$ , the flow is unstable with a  $c_i > 0$ .

Physically,  $Fr_t^2 \leq 1$  implies that the total kinetic energy is smaller than the total potential energy. So the potential energy might transfer to the kinetic energy after being disturbed, and the flow becomes unstable. On the other hand, when the potential energy is smaller than the kinetic energy ( $Fr_t^2 > 1$ ), the flow is stable because no potential energy could transfer to the kinetic energy.

Mathematically, we need derive some useful formula for applications, since there is still unknown  $c_r$  in above equations. To this aim, we rewrite Eq.(2.9) as

$$h(y) = \mu(U + \frac{U''}{2\mu} - c_r)^2 - (N^2 + \frac{U''^2}{4\mu}). \quad (2.11)$$

Assume that the minimum and maximum value of  $U + \frac{U''}{2\mu}$  within  $a \leq y \leq b$  is respectively  $m_i$  and  $m_a$ . It is from Eq.(2.11) that  $m_i \leq c_r \leq m_a$  for the smallest value of  $h(y)$ . Thus a general theorem for instability can be obtained from the above notations.

Theorem 2: If velocity  $U$  and stable stratification  $N^2$  satisfy  $h(y) \leq 0$  throughout the domain for a certain  $m_i \leq c_r \leq m_a$ , there must be a  $c_i > 0$  and the flow is unstable.

It is from Eq.(2.11) that  $h(y) \leq 0$  requires  $\mu(U + \frac{U''}{2\mu} - c_r)^2$  less than  $(N^2 + \frac{U''^2}{4\mu})$ . The bigger  $N^2$  is, the smaller  $h(y)$  is. So the stable stratification has a destabilization mechanism in shear flow. This conclusion is new as former theoretic studies always took the static stable stratification as the stable effects for shear flows.

According to Eq.(2.11), the bigger  $m_a - m_i$  is, the more stable the flow is. It is obvious that  $m_a - m_i$  is bigger for  $U'/U''' > 0$  than that for  $U'/U''' < 0$ . So the flow is more stable with the velocity profile  $U'/U''' > 0$ .

Although Theorem 1 gives a sufficient unstable condition for instability, the complicated expression makes it difficult for application. In the following section we will derive simple and useful criteria.

### 3. Criteria For Flow Instability

#### 3.1. Inviscid Flow

The simplest flow is the inviscid shear flow with  $N^2 = 0$ . The sufficient condition for instability is  $h(y) \leq 0$ . To find such condition, we rewrite  $h(y)$  in Eq.(2.11) as

$$h(y) = (U_1 - c_r)(U_2 - c_r) \quad (3.1)$$

where  $U_1 = U$  and  $U_2 = U + U''/\mu$ . Then there may be three cases. Two of them have  $U_1$  intersecting with  $U_2$  at  $U_s'' = 0$  (Fig.1). The first case is that  $U''/(U - U_s) > 0$ ; thus,  $h(y) > 0$  always holds at  $c_r = U_s$  as shown in Fig.1a. The second case is that  $U''/(U - U_s) < 0$ ; thus,  $h(y) < 0$  always holds in the whole domain, as shown in Fig.1b. In this case, the flow might be unstable.

The sufficient condition for instability can be found from Eq.(3.1) as shown in Fig.1b. Given  $c_r = U_s$ , Eq.(3.1) becomes

$$h(y) = \frac{(U - U_s)^2}{\mu} [\mu + \frac{U''}{(U - U_s)}] \quad (3.2)$$

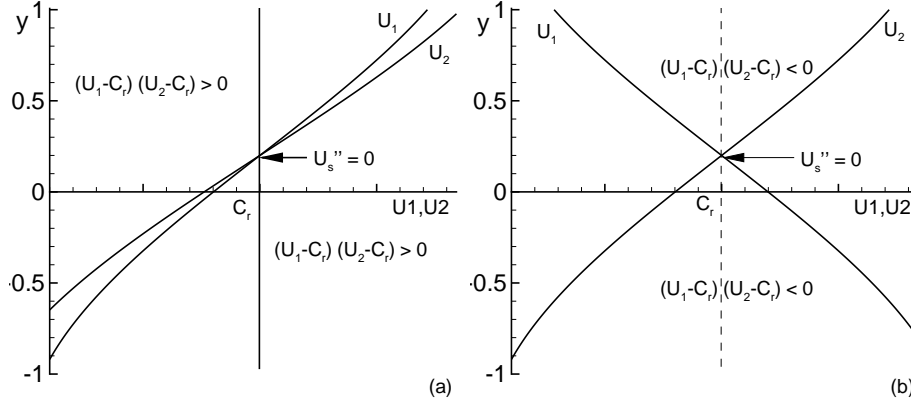


FIGURE 1. The value of  $h(y)$  under the condition  $U_s'' = 0$ : (a) for  $U''/(U - U_s) > 0$ , (b) for  $U''/(U - U_s) < 0$ .

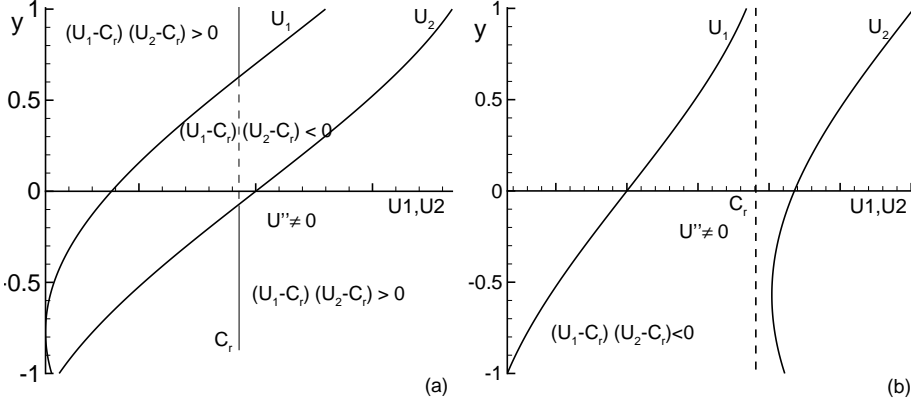


FIGURE 2. The value of  $h(y)$  for  $U'' \neq 0$  in case 3 and case 4.

If  $\frac{U''}{(U - U_s)} < -\mu$  is always satisfied,  $h(y) < 0$  holds within the domain.

Corollary 1.1: If the velocity profile satisfies  $\frac{U''}{U - U_s} < -\mu$  within the domain, the flow is unstable.

Since Sun (2007) obtained a sufficient condition for stability, i.e.  $\frac{U''}{U - U_s} > -\mu$  within the domain. The above condition for instability is nearly marginal (Sun 2008).

The last case is that  $U'' \neq 0$  throughout the domain; thus,  $h(y) > 0$  always exists somewhere within the domain, as shown in Fig.2a.

### 3.2. Stably Stratified Flow

If the static stratification is stable ( $N^2 > 0$ ), then  $g(y)$  is positive. The flow is unstable if  $h(y)$  is negatively defined within  $a \leq y \leq b$  at  $k = 0$ . We rewrite  $h(y)$  as

$$\begin{aligned} h(y) &= \mu(U_1 - c_r)(U_2 - c_r) \\ &= \mu \left[ U + \frac{1}{2\mu} (U'' - \sqrt{U''^2 + 4\mu N^2}) - c_r \right] \\ &\quad \times \left[ U + \frac{1}{2\mu} (U'' + \sqrt{U''^2 + 4\mu N^2}) - c_r \right]. \end{aligned} \quad (3.3)$$

The value of  $h(y)$  can be classified into 4 cases. The first and the second ones ( $U_s'' = 0$  and  $N_s^2 = 0$  at  $y = y_s$ ) are similar to discussed above and shown in Fig1a and Fig1b. For

such cases, we have a sufficient condition for instability,

$$\frac{U''(U - U_s) - N^2}{(U - U_s)^2} < -\mu. \quad (3.4)$$

This can be derived directly from Eq.(2.7), similar to Corollary 1.1. The first sufficient condition for instability is due to the shear instability, and the unstable criterion is Eq.(3.4).

Corollary 1.2: If the velocity profile satisfies  $\frac{U''(U - U_s) - N^2}{(U - U_s)^2} < -\mu$  within the domain, the flow is unstable.

The third case ( $U'' \neq 0$ ) is also similar to the case in Fig2a, and the flow is stable. The last one is unstable flow shown in Fig.2b, where  $U'' \neq 0$  and  $h(y) < 0$  throughout. In the last case, the maximum of  $U_1$  must be smaller than the minimum of  $U_2$  so that a proper  $c_r$  within the  $U_1$  and  $U_2$  could be used for the unstable waves. Although the exact criterion can not be obtained as the required maximum and minimum can not be explicitly given, the approach is very straightforward.

Nevertheless, we can also obtain some approximate criterion for the fourth case. It is from Eq.(2.11) that  $h(y) \leq 0$  if the minimax of  $\mu(U + \frac{U''}{2\mu} - c_r)^2$  is less than the minimum of  $(N^2 + \frac{U''^2}{4\mu})$ . As the minimax value of  $\mu(U + \frac{U''}{2\mu} - c_r)^2$  is  $\frac{1}{4}\mu(m_a - m_i)^2$  when  $c_r = (m_a + m_i)/2$ , we obtained a new criterion according to Eq.(2.10).

$$Fr_t^2(c_r) = \frac{1}{4\mu N^2} [\mu^2(m_a + m_i)^2 - U''^2]. \quad (3.5)$$

Thus a sufficient (but not necessary) condition for  $h(y) < 0$  is that the following equation holds for  $a \leq y \leq b$ .

$$Fr_t^2 \leq 1. \quad (3.6)$$

From the above corollaries, the flow might be unstable if the static stable stratification is strong enough. The stably stratification destabilize the flow, which is a new unstable mechanism. The above corollary contradicts the previous results (Abarbanel *et al.* 1984), but it agrees well with the recent theory (Friedlander 2001), experiments (Zilitinkevich *et al.* 2008) and simulations (Alexakis 2009). Again, we point out here that the flow is unstable due to potential energy transfer to kinetic energy under the condition of  $Fr_t^2 \leq 1$ .

This conclusion is new because it is quite different from previous theorems in which the static stable stratification plays the role as a stabilizing factor for shear flows.

## 4. Discussion

### 4.1. Necessary Instability Criterion

In the above investigation, it was found that stable stratification is a destabilization mechanism for the flow. Such finding is not surprising if one notes the terms in Eq.(2.2). Mathematically the sum of terms in square brackets should be negative for the wave solution. Thus both  $\frac{U''}{U - c} > 0$  and  $N^2 < 0$  do favor this condition. This is why the unstable solutions always occur at  $\frac{U''}{U - c} < 0$  in shear flow. And  $N^2 > 0$  here might lead to  $c_i^2 > 0$ . Physically, the perturbation waves are truncated in the neutral stratified flow. But the stable stratification allows wide range of waves in the perturbation. Such waves might interact with each other like what was illustrated in Sun (2008).

As Theorem 1 is the only sufficient condition, it is hypothesized that the criterion is not only the sufficient but also the necessary condition for instability in stably stratified flow. This hypothesis might be criticized in that the flow might be unstable ( $c_i^2 > 0$ )

if  $h(y)$  changes sign within the interval (Fig2a), where a proper chosen  $\phi$  would let the right hand of Eq.(2.7) become negative.

However, this criticism is not valid for the case in Fig2a. It is from the well-known criteria (e.g. Rayleigh's inflexion point theorem) that the proper chosen  $\phi$  always let the right hand of Eq.(2.7) vanish. It seems that the flow tends to be stable, or the perturbations have a prior policy to let  $c_i = 0$ . The flow become unstable unless any choice of  $\phi$  would let the right hand of Eq.(2.7) be negative. In this situation, we hypothesize that Theorem 1 fully solves the stability problem.

#### 4.2. Long-wave Instability

In inviscid shear flows, it has been recognized that very short-wave perturbations are dynamically stable under neutral stratification, and the dynamic instability is due to the larger wavelengths (Sun 2006b). It should be noted that Rayleigh's case is reduced to the Kelvin-Helmholtz vortex sheet model under the long-wave limit  $k \ll 1$  (Huerre & Rossi 1998; Criminale *et al.* 2003). We have shown that this can be extended to shear flows, and that the growth rate  $\omega_i$ , is proportional to  $\sqrt{\mu_1}$  (Sun 2006b, 2008).

Such conclusion can be simply generated to the stratified shear flows, which can be seen from Eq.(2.8). If  $k$  is larger than a critical value  $k_c$ , the sufficient condition in Theorem 1 can not be satisfied and the flow is stable. For shortwave ( $k \gg 1$ ),  $h(y)$  is always larger than that for long-wave  $k \ll 1$ . The long-wave instability in the stratified shear flow was also noted by Miles (1961, 1963) and Howard (1961), who showed a likelihood of  $c_i \rightarrow 0$  at  $k \rightarrow \infty$ . The long-wave instability theory can explain the results in numerical simulations (Alexakis 2009), where the unstable perturbations are long-wave.

#### 4.3. Local Criterion

In the above investigations, an parameter  $\mu$  is used, which represents the ratio of two integrations with boundaries. So the criteria are global. On the other hand, we can also investigate the local balance without boundary conditions. For example, consider the flow within a thick layer  $-\delta \leq y \leq \delta$ . The velocity is  $U(y) = U_0 + U'y$ , and the kinetic energy is  $(U - c_r)^2$ . The stratification is  $N^2$ , and the potential energy is  $N^2 d^2$ , where  $d = 2\delta$  is the thickness of the layer. The Froude number is  $Fr_t^2 = (U'^2 \delta^2)/(N^2 d^2)$  for  $c_r = U_0$ . The instability criterion in Eq.(3.6) becomes

$$Ri = \frac{U'^2}{N^2} > \frac{1}{4}. \quad (4.1)$$

If local gradient Richardson number exceeds  $1/4$ , the local disturbances is unstable. However, the flow might be stable as the globe total Froude number  $Fr_t^2 > 1$ . This criterion is opposite to Miles-Howard theorem, we will show why Miles-Howard' theorem is not correct from their derivations.

#### 4.4. Relations to Other Theories

In the inviscid shear flow, the linear theories, e.g., Rayleigh-Kuo criterion (Criminale *et al.* 2003), Fjortoft criterion (Fjortoft 1950) and Sun's criterion (Sun 2007), are equal to Arnol'd's nonlinear stability criterion (Arnold 1965). Arnol'd's first stability theorem corresponds to Fjortoft's criterion (Drazin & Reid 2004; Criminale *et al.* 2003), and Arnol'd's second nonlinear theorem corresponds to Sun's criterion (Sun 2007, 2008). It is obvious that the present theory, especially Corollary 1.1 is a natural generalization of inviscid theories.

In the stratified flow, Miles (1961, 1963) and Howard (1961) applied a transform  $F =$

$\phi/(U - c)^n$  to Eq.(2.2), which allows different kind of perturbations. Thus  $n = 1/2$  gives Miles's theory and  $n = 1$  gives Howard's semicircle theorem.

Considering that  $n = 1$  and  $N^2 = 0$  (Howard 1961), Eq.(2.4) becomes

$$\int_a^b (|F'|^2 + k^2|F|^2)[(U - c_r)^2 - c_i^2]dy = 0. \quad (4.2)$$

It is from Eq.(4.2) that all the inviscid flows (no matter what the velocity profile  $U(y)$  is) must be temporal unstable ( $k$  is real). This contradicts the criteria (both linear and nonlinear ones) for inviscid shear flow. So the wavenumber  $k$  in Eq.(4.2) should be complex  $k = k_r + ik_i$ . Besides, from Eq.(2.11), Eq.(3.3) and Fig.2b, the unstable  $c_r$  might be either within the value of  $U$  or beyond the value of  $U$ . This also contradicts the Howard's semicircle theorem for the stratified flow. It implies that the transform  $F$  is not suitable for temporal stability problem.

Taking  $n = 1/2$ , Howard extracted a new equation from Taylor-Goldstein equation,

$$[(U - c)F']' - [k^2(U - c) + \frac{U''}{2} + (\frac{1}{4}U'^2 - N^2)/(U - c)]F = 0 \quad (4.3)$$

After multiplying above equation by the complex conjugate of  $F$  and integrating over the flow regime, then the imaginary part of the expression is

$$-c_i \int_a^b |F'|^2 + [k^2|F|^2 + (\frac{1}{4}U'^2 - N^2)|F|^2/|U - c|^2] = 0 \quad (4.4)$$

Miles-Howard theorem concludes that if  $c_i \neq 0$ , then  $Ri < \frac{1}{4}$  for instability.

However, the transform  $F = \phi/\sqrt{U - c}$  requires a complex function  $F$ , even though both  $\phi$  and  $c$  are real. In that  $\sqrt{U - c}$  might be complex somewhere as  $U - c_r < 0$ . Consequently, the wave number  $k$  in Eq.(4.3) is a complex number but no longer a real number as that assumed in Taylor-Goldstein equation. The complex wavenumber  $k$  leads to spatial stability problem but temporal stability problem investigated in this study. The assumption of  $c_i = 0$  with  $k_i \neq 0$  implies the flow is unstable with  $\omega_i \neq 0$ . However, Howard ignored this in his derivations. That is why Miles-Howard theorem leads contradictions to the present studies.

Although the transform  $F = \phi/(U - c)^n$  leads some contradictions with Rayleigh criterion and present results, it might be useful for the viscous flows. In these flows, the spatial but temporal stability problem is dominated, and  $k = k_r + ik_i$  is the complex wavenumber. It is well known that the plane Couette flow is viscously unstable for Reynolds number  $Re > Re_c$  from the experiments but viscously stable from the Orr-Sommerfeld equation (Criminale *et al.* 2003). If applying the transform in Eq.(4.2) all the inviscid flows must be unstable. Thus the plane Couette flow might be stable only for  $Re < Re_c$  due to the stabilization of the viscosity.

It is argued that the Taylor-Goldstein equation represents temporal instability, the transform represents spatial instability (Huerre & Rossi 1998; Criminale *et al.* 2003). In that the perturbation is seen along with the flow at the speed of  $(U - c)$  in Miles (1961); Howard (1961). The transform  $F = \phi/(U - c)^n$  also turns real wavenumber  $k$  into complex number,  $c_i = 0$  implies  $\omega_i \neq 0$ . The assumption of real  $k$  after transform will leads to contradictions with the results derived from the Taylor-Goldstein equation. So the previous investigators can hardly generalize their results from homogeneous fluids to stratified fluids.



## 5. Conclusion

In summary, the stably stratification is a destabilization mechanism, and the flow instability is due to the competition of the kinetic energy with the potential energy. Globally, the flow is always unstable when the total Froude number  $Fr_t^2 \leq 1$ , where the larger potential energy might transfer to the kinetic energy after being disturbed. Locally, the flow is unstable as the gradient Richardson number  $Ri > 1/4$ . The approach is very straightforward and can be used for similar analysis. In the inviscid stratified flow, the unstable perturbation must be long-wave scale. This result extends the Rayleigh's, Fjørtoft's, Sun's and Arnol'd's criteria for the inviscid homogenous fluid, but contradicts the well-known Miles and Howard theorems. It is argued here that the transform  $F = \phi/(U - c)^n$  is not suitable for temporal stability problem, and that it will leads to contradictions with the results derived from Taylor-Goldstein equation.

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